STRATIFIED CRITICAL POINTS ON THE REAL MILNOR FIBRE AND INTEGRAL-GEOMETRIC FORMULAS

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Dedicated to professor David Trotman on his 60th birthday

Abstract. Let $(X,0) \subset (\mathbb{R}^n,0)$ be the germ of a closed subanalytic set and let f and $g : (X,0) \to (\mathbb{R},0)$ be two subanalytic functions. Under some conditions, we relate the critical points of g on the real Milnor fibre $X \cap f^{-1}(\delta) \cap B_\epsilon$, $0 < |\delta| \ll \epsilon \ll 1$, to the topology of this fibre and other related subanalytic sets. As an application, when g is a generic linear function, we obtain an "asymptotic" Gauss-Bonnet formula for the real Milnor fibre of f. From this Gauss-Bonnet formula, we deduce "infinitesimal" linear kinematic formulas.

1. Introduction

Let $F = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, $2 \leq k \leq n$, be a complete intersection with isolated singularity. The Lê-Greuel formula [21, 22] states that

$$\mu(F') + \mu(F) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{I},$$

where $F' : (\mathbb{C}^n, 0) \to (\mathbb{C}^{k-1}, 0)$ is the map with components $f_1, \ldots, f_{k-1}$, I is the ideal generated by $f_1, \ldots, f_{k-1}$ and the $(k \times k)$-minors $\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_k})}$ and $\mu(F)$ (resp. $\mu(F')$) is the Milnor number of F (resp F'). Hence the Lê-Greuel formula gives an algebraic characterization of a topological data, namely the sum of two Milnor numbers. However, since the right-hand side of the above equality is equal to the number of critical points of f_k, counted with multiplicity, on the Milnor fibre of F', the Lê-Greuel formula can be also viewed as a topological characterization of this number of critical points.

Many works have been devoted to the search of a real version of the Lê-Greuel formula. Let us recall them briefly. We consider an analytic map-germ $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $2 \leq k \leq n$, and we denote by F' the map-germ $(f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)$. Some authors investigated the following difference:

$$D_{\delta,\delta'} = \chi\big(F'^{-1}(\delta) \cap \{f_k \geq \delta'\} \cap B_\epsilon\big) - \chi\big(F'^{-1}(\delta) \cap \{f_k \leq \delta'\} \cap B_\epsilon\big),$$

where (δ, δ') is a regular value of F such that $0 \leq |\delta'| \ll |\delta| \ll \epsilon$.

In [12], we proved that

$$D_{\delta,\delta'} \equiv \dim_{\mathbb{R}} \frac{\mathcal{O}_{\mathbb{R}^n,0}}{I} \bmod 2,$$

where $\mathcal{O}_{\mathbb{R}^n,0}$ is the ring of analytic function-germs at the origin and I is the ideal generated by $f_1, \ldots, f_{k-1}$ and all the $k \times k$ minors $\frac{\partial(f_k, f_1, \ldots, f_{k-1})}{\partial(x_{i_1}, \ldots, x_{i_k})}$. This is only a mod 2 relation and we may ask if it is possible to get a more precise relation.

When $k = n$ and $f_k = x_1^2 + \cdots + x_n^2$, according to Aoki et al. ([1], [3]), $D_{\delta,0} = \chi\big(F'^{-1}(\delta) \cap B_\varepsilon\big) = 2\deg_0 H$ and $2\deg_0 H$ is the number of semi-branches of $F'^{-1}(0)$, where

$$H = \Big(\frac{\partial(f_n, f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_n)}, f_1, \ldots, f_{n-1}\Big).$$

They proved a similar formula in the case $f_k = x_n$ in [2] and Szafraniec generalized all these results to any f_k in [23].

When $k = 2$ and $f_2 = x_1$, Fukui [18] stated that

$$D_{\delta,0} = -\mathrm{sign}(-\delta)^n \deg_0 H,$$

where $H = (f_1, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n})$. Several generalizations of Fukui's formula are given in [19], [11], [20] and [13].

In all these papers, the general idea is to count algebraically the critical points of a Morse perturbation of f_k on $F'^{-1}(\delta) \cap B_\epsilon$ and to express this sum in two ways: as a difference of Euler characteristics and as a topological degree. Using the Eisenbud-Levine formula [16], this latter degree can be expressed as a signature of a quadratic form and so, we obtain an algebraic expression for $D_{\delta,\delta'}$.

In this paper, we give a real and stratified version of the Lê-Greuel formula. We restrict ourselves to the topological aspect and relate a sum of indices of critical points on a real Milnor fibre to some Euler characteristics (this is also the point of view adopted in [7]). More precisely, we consider a germ of a closed subanalytic set $(X, 0) \subset (\mathbb{R}^n, 0)$ and a subanalytic function $f : (X, 0) \to (\mathbb{R}, 0)$. We assume that X is contained in a open set U of $\mathbb{R}^n$ and that f is the restriction to X of a C^2-subanalytic function $F : U \to \mathbb{R}$. We denote by X^f the set $X \cap f^{-1}(0)$ and we equip X with a Thom stratification adapted to X^f. If $0 < |\delta| \ll \epsilon \ll 1$ then the real Milnor fibre of f is defined by

$$M_f^{\delta,\epsilon} = f^{-1}(\delta) \cap X \cap B_\epsilon.$$

We consider another subanalytic function $g : (X, 0) \to (\mathbb{R}, 0)$ and we assume that it is the restriction to X of a C^2-subanalytic function $G : U \to \mathbb{R}$. We denote by X^g the intersection $X \cap g^{-1}(0)$. Under two conditions on g, we study the topological behaviour of $g_{|M_f^{\delta,\epsilon}}$.

We recall that if $Z \subset \mathbb{R}^n$ is a closed subanalytic set, equipped with a Whitney stratification and $p \in Z$ is an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, restriction to Z of a C^2-subanalytic function Φ, then the index of ϕ at p is defined as follows:

$$\mathrm{ind}(\phi, Z, p) = 1 - \chi\big(Z \cap \{\phi = \phi(p) - \eta\} \cap B_\epsilon(p)\big),$$

where $0 < \eta \ll \epsilon \ll 1$ and $B_\epsilon(p)$ is the closed ball of radius ϵ centered at p. Let $p_1^{\delta,\epsilon}, \ldots, p_r^{\delta,\epsilon}$ be the critical points of g on $X \cap f^{-1}(\delta) \cap \mathring{B}_\epsilon$, where $\mathring{B}_\epsilon$ denotes the open ball of radius ϵ. We set

$$I(\delta,\epsilon,g) = \sum_{i=1}^{r} \mathrm{ind}(g, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta,\epsilon,-g) = \sum_{i=1}^{r} \mathrm{ind}(-g, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

Our main theorem (Theorem 3.10) is the following:

$$I(\delta,\epsilon,g)+I(\delta,\epsilon,-g) = 2\chi(M_f^{\delta,\epsilon})-\chi(X\cap f^{-1}(\delta)\cap S_\epsilon)-\chi(X^g\cap f^{-1}(\delta)\cap S_\epsilon).$$

As a corollary (Corollary 3.11), when $f : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, we obtain that

$$I(\delta,\epsilon,g) + I(\delta,\epsilon,-g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(\mathrm{Lk}(X^f)) - \chi(\mathrm{Lk}(X^f \cap X^g)),$$

where $\mathrm{Lk}(-)$ denotes the link at the origin.

Then we apply these results when g is a generic linear form to get an asymptotic Gauss-Bonnet formula for $M_f^{\delta,\epsilon}$ (Theorem 4.5). In the last section, we use this asymptotic Gauss-Bonnet formula to prove infinitesimal linear kinematic formulas for closed subanalytic germs (Theorem 5.5), that generalize the Cauchy-Crofton formula for the density due to Comte [8].

The paper is organized as follows. In Section 2, we prove several lemmas about critical points on the link of a subanalytic set. Section 3 contains real stratified versions of the Lê-Greuel formula. In Section 4, we establish the asymptotic Gauss-Bonnet formula and in Section 5, the infinitesimal linear kinematic formulas.

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2. Lemmas on critical points on the link of a stratum

In this section, we study the behaviour of the critical points of a C^2-subanalytic function on the link of stratum that contains 0 in its closure, for a generic choice of the distance function to the origin.

Let $Y \subset \mathbb{R}^n$ be a C^2-subanalytic set such that 0 belongs to its closure $\overline{Y}$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a C^2-subanalytic function such that $\theta(0) = 0$. We will first study the behaviour of the critical points of $\theta_{|Y} : Y \to \mathbb{R}$ in the neighborhood of 0, and then the behaviour of the critical points of the restriction of θ to the link of 0 in Y.

Lemma 2.1. *The critical points of $\theta_{|Y}$ lie in $\{\theta = 0\}$ in a neighborhood of 0.*

Proof. By the Curve Selection Lemma, we can assume that there is a C^1-subanalytic curve $\gamma : [0, \nu[\to \overline{Y}$ such that $\gamma(0) = 0$ and $\gamma(t)$ is a critical point of $\theta_{|Y}$ for $t \in]0, \nu[$. Therefore, we have

$$(\theta \circ \gamma)'(t) = \langle \nabla \theta_{|Y}(\gamma(t)), \gamma'(t) \rangle = 0,$$

since $\gamma'(t)$ is tangent to Y at $\gamma(t)$. This implies that $\theta \circ \gamma(t) = \theta \circ \gamma(0) = 0$. $\square$

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be another C^2-subanalytic function such that $\rho^{-1}(a)$ intersects Y transversally. Then the set $Y \cap \{\rho \leq a\}$ is a manifold with boundary. Let p be a critical point of $\theta_{|Y \cap \{\rho \leq a\}}$ which lies in $Y \cap \{\rho = a\}$ and which is not a critical point of $\theta_{|Y}$. This implies that

$$\nabla \theta_{|Y}(p) = \lambda(p) \nabla \rho_{|Y}(p),$$

with $\lambda(p) \neq 0$.

Definition 2.2. *We say that $p \in Y \cap \{\rho = a\}$ is an outwards-pointing (resp. inwards-pointing) critical point of $\theta_{|Y \cap \{\rho \leq a\}}$ if $\lambda(p) > 0$ (resp. $\lambda(p) < 0$).*

Now let us assume that $\rho : \mathbb{R}^n \to \mathbb{R}$ is a distance function to the origin which means that $\rho \geq 0$ and $\rho^{-1}(0) = \{0\}$ in a neighborhood of 0. By Lemma 2.1, we know that for $\epsilon > 0$ small enough, the level $\rho^{-1}(\epsilon)$ intersects Y transversally. Let p^ϵ be a critical point of $\theta_{|Y \cap \rho^{-1}(\epsilon)}$ such that $\theta(p^\epsilon) \neq 0$. This means that there exists $\lambda(p^\epsilon)$ such that

$$\nabla \theta_{|Y}(p^\epsilon) = \lambda(p^\epsilon) \nabla \rho_{|Y}(p^\epsilon).$$

Note that $\lambda(p^\epsilon) \neq 0$ because $\nabla \theta_{|Y}(p^\epsilon) \neq 0$ for $\theta(p^\epsilon) \neq 0$.

Lemma 2.3. *The point p^ϵ is an outwards-pointing (resp. inwards-pointing) for $\theta_{|Y \cap \{\rho \leq \epsilon\}}$ if and only if $\theta(p^\epsilon) > 0$ (resp. $\theta(p^\epsilon) < 0$).*

Proof. Let us assume that $\lambda(p^\epsilon) > 0$. By the Curve Selection Lemma, there exists a C^1-subanalytic curve $\gamma : [0, \nu[\to \overline{Y}$ passing through p^ϵ such that $\gamma(0) = 0$ and for $t \neq 0$, $\gamma(t)$ is a critical point of $\theta_{|Y \cap \{\rho = \rho(\gamma(t))\}}$ with $\lambda(\gamma(t)) > 0$. Therefore we have

$$(\theta \circ \gamma)'(t) = \langle \nabla \theta_{|Y}(\gamma(t)), \gamma'(t) \rangle = \lambda(\gamma(t)) \langle \nabla \rho_{|Y}(\gamma(t)), \gamma'(t) \rangle.$$

But $(\rho \circ \gamma)' > 0$ for otherwise $(\rho \circ \gamma)' \leq 0$ and $\rho \circ \gamma$ would be decreasing. Since $\rho(\gamma(t))$ tends to 0 as t tends to 0, this would imply that $\rho \circ \gamma(t) \leq 0$, which is impossible. We can conclude that $(\theta \circ \gamma)' > 0$ and that $\theta \circ \gamma$ is strictly increasing. Since $\theta \circ \gamma(t)$ tends to 0 as t tends to 0, we see that $\theta \circ \gamma(t) > 0$ for $t > 0$. Similarly if $\lambda(p^\epsilon) < 0$ then $\theta(p^\epsilon) < 0$. $\square$

Now we will study these critical points for a generic choice of the distance function. We denote by $\mathrm{Sym}(\mathbb{R}^n)$ the set of symmetric $n \times n$-matrices with real entries, by $\mathrm{Sym}^*(\mathbb{R}^n)$ the open dense subset of such matrices with non-zero determinant and by $\mathrm{Sym}^{*,+}(\mathbb{R}^n)$ the open subset of these invertible matrices that are positive definite or negative definite. Note that these sets are semi-algebraic. For each $A \in \mathrm{Sym}^{*,+}(\mathbb{R}^n)$, we denote by ρ_A the following quadratic form:

$$\rho_A(x) = \langle Ax, x \rangle.$$

We denote by $\Gamma^Y_{\theta,A}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,A} = \left\{ x \in Y \mid \mathrm{rank} \left[\nabla \theta_{|Y}(x), \nabla \rho_{A|Y}(x) \right] < 2 \right\},$$

and by Σ^Y_θ the set of critical points of $\theta_{|Y}$. Note that $\Sigma^Y_\theta \subset \{\theta = 0\}$ by Lemma 2.1.

Lemma 2.4. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,A} \setminus (\Sigma^Y_\theta \cup \{0\})$ is a C^1-subanalytic curve (possible empty) in a neighborhood of 0.*

Proof. We can assume that $\dim Y > 1$. Let

$$Z = \Big\{ (x, A) \in \mathbb{R}^n \times \mathrm{Sym}^{+,*}(\mathbb{R}^n) \mid x \in Y \setminus (\Sigma^Y_\theta \cup \{0\}) \\ \text{and } \mathrm{rank} \left[\nabla \theta_{|Y}(x), \nabla \rho_{A|Y}(x) \right] < 2 \Big\}.$$

Let (y, B) be a point in Z. We can suppose that around y, Y is defined by the vanishing of k subanalytic functions $f_1, \ldots, f_k$ of class C^2. Hence in a neighborhood of (y, B), Z is defined be the vanishing of $f_1, \ldots, f_k$ and the minors

$$\frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_{i_1}, \ldots, x_{i_{k+2}})}.$$

Furthermore, since y does not belong to Σ^Y_θ, we can assume that

$$\frac{\partial(f_1, \ldots, f_k, \theta)}{\partial(x_1, \ldots, x_k, x_{k+1})} \neq 0,$$

in a neighborhood of y. Therefore Z is locally defined by $f_1 = \cdots = f_k = 0$ and

$$\frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_1, \ldots, x_{k+1}, x_{k+2})} = \cdots = \frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_1, \ldots, x_{k+1}, x_n)} = 0.$$

Let us write $M = \frac{\partial(f_1,\ldots,f_k,\theta)}{\partial(x_1,\ldots,x_k,x_{k+1})}$ and for $i \in \{k+2, \ldots, n\}$, $m_i = \frac{\partial(f_1,\ldots,f_k,\theta,\rho_A)}{\partial(x_1,\ldots,x_{k+1},x_i)}$. If $A = [a_{ij}]$ then

$$\rho_A(x) = \sum_{i=1}^n a_{ii} x_i^2 + 2 \sum_{i \neq j} a_{ij} x_i x_j,$$

and so $\frac{\partial \rho_A}{\partial x_i}(x) = 2 \sum_{j=1}^n a_{ij} x_j$. For $i \in \{k+1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$, we have

$$\frac{\partial m_i}{\partial a_{ij}} = 2 x_j M.$$

Since $y \neq 0$, one of the x_j's does not vanish in the neighborhood of y and we can conclude that the rank of

$$[\nabla f_1(x), \ldots, \nabla f_k(x), \nabla m_{k+2}(x, A), \ldots, \nabla m_n(x, A)]$$

is $n-1$ and that Z is a C^1-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1$. Now let us consider the projection $\pi_2 : Z \to \mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $(x, A) \mapsto A$. Bertini-Sard's theorem implies that the set D_{π_2} of critical values of π_2 is a subanalytic set of dimension strictly less than $\frac{n(n+1)}{2}$. Hence, for all $A \notin D_{\pi_2}$, $\pi_2^{-1}(A)$ is a C^1-subanalytic curve (possibly empty). But this set is exactly $\Gamma^Y_{\theta,A} \setminus (\Sigma^Y_\theta \cup \{0\})$. □

Let $R \subset Y$ be a subanalytic set of dimension strictly less than $\dim Y$. We will need the following lemma.

Lemma 2.5. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,A} \setminus (\Sigma^Y_\theta \cup \{0\}) \cap R$ is a subanalytic set of dimension at most 0 in a neighborhood of 0.*

Proof. Let us put $l = \dim Y$. Since R admits a locally finite subanalytic stratification, we can assume that R is a C^2-subanalytic manifold of dimension d with $d < l$. Let W be the following subanalytic set:

$$W = \Big\{(x, A) \in \mathbb{R}^n \times \mathrm{Sym}^{+,*}(\mathbb{R}^n) \mid x \in R \setminus (\Sigma^Y_\theta \cup \{0\}) \\ \text{and rank} \Big[\nabla \theta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 2\Big\}.$$

Using the same method as in the previous lemma, we can prove that W is a C^1-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1 + d - l$ and conclude, remarking that $d - l \leq -1$. □

Now we introduce a new C^2-subanalytic function $\beta : \mathbb{R}^n \to \mathbb{R}$ such that $\beta(0) = 0$. We denote by $\Gamma^Y_{\theta,\beta,A}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,\beta,A} = \Big\{x \in Y \mid \text{rank} \Big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 3\Big\},$$

and by $\Gamma^Y_{\theta,\beta}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,\beta} = \big\{x \in Y \mid \text{rank} \big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x)\big] < 2\big\}.$$

Lemma 2.6. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,\beta,A} \setminus (\Gamma^Y_{\theta,\beta} \cup \{0\})$ is a C^1-subanalytic set of dimension at most 2 (possibly empty) in a neighborhood of 0.*

Proof. We can assume that $\dim Y > 2$. Let

$$Z = \Big\{(x, A) \in \mathbb{R}^n \times \mathrm{Sym}^{+,*}(\mathbb{R}^n) \mid x \in Y, \text{rank} \big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x)\big] = 2 \\ \text{and rank} \Big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 3\Big\}.$$

Let (y, B) be a point in Z. We can suppose that around y, Y is defined by the vanishing of k subanalytic functions $f_1, \ldots, f_k$ of class C^2. Hence in a neighborhood of (y,B), Z is defined by the vanishing of $f_1,\dots,f_k$ and the minors

$$\frac{\partial(f_1,\dots,f_k,\theta,\beta,\rho_A)}{\partial(x_{i_1},\dots,x_{i_{k+3}})}.$$

Since y does not belong to $\Gamma^Y_{\theta,\beta}$, we can assume that

$$\frac{\partial(f_1,\dots,f_k,\theta,\beta)}{\partial(x_1,\dots,x_k,x_{k+1},x_{k+2})}\neq 0,$$

in a neighborhood of y. Therefore Z is locally defined by $f_1,\dots,f_k=0$ and

$$\frac{\partial(f_1,\dots,f_k,\theta,\beta,\rho_A)}{\partial(x_1,\dots,x_{k+2},x_{k+3})}=\cdots=\frac{\partial(f_1,\dots,f_k,\theta,\beta,\rho_A)}{\partial(x_1,\dots,x_{k+2},x_n)}=0.$$

It is clear that we can apply the same method as Lemma 2.4 to get the result. □

3. Lê-Greuel type formula

In this section, we prove the Lê-Greuel type formula announced in the introduction.

Let $(X,0)\subset(\mathbb{R}^n,0)$ be the germ of a closed subanalytic set and let $f:(X,0)\to(\mathbb{R},0)$ be a subanalytic function. We assume that X is contained in a open set U of $\mathbb{R}^n$ and that f is the restriction to X of a C^2-subanalytic function $F:U\to\mathbb{R}$. We denote by X^f the set $X\cap f^{-1}(0)$ and by [4], we can equip X with a Thom stratification $\mathcal{V}=\{V_\alpha\}_{\alpha\in A}$ adapted to X^f. This means that $\{V_\alpha\in\mathcal{V}\mid V_\alpha\not\subseteq X^f\}$ is a Whitney stratification of $X\setminus X^f$ and that for any pair of strata (V_α,V_β) with $V_\alpha\not\subseteq X^f$ and $V_\beta\subset X^f$, the Thom condition is satisfied.

Let us denote by $\Sigma_{\mathcal{V}}f$ the critical locus of f. It is the union of the critical loci of f restricted to each stratum, i.e. $\Sigma_{\mathcal{V}}f=\cup_\alpha\Sigma(f_{|V_\alpha})$, where $\Sigma(f_{|V_\alpha})$ is the critical set of $f_{|V_\alpha}:V_\alpha\to\mathbb{R}$. Since $\Sigma_{\mathcal{V}}f\subset f^{-1}(0)$ (see Lemma 2.1), the fibre $f^{-1}(\delta)$ intersects the strata V_α's, $V_\alpha\not\subseteq X^f$, transversally if δ is sufficiently small. Hence it is Whitney stratified with the induced stratification $\{f^{-1}(\delta)\cap V_\alpha\mid V_\alpha\not\subseteq X^f\}$.

By Lemma 2.1, we know that if $\epsilon>0$ is sufficiently small then the sphere S_ϵ intersects X^f transversally. By the Thom condition, this implies that there exists $\delta(\epsilon)>0$ such that for each δ with $0<|\delta|\leq\delta(\epsilon)$, the sphere S_ϵ intersects the fibre $f^{-1}(\delta)$ transversally as well. Hence the set $f^{-1}(\delta)\cap B_\epsilon$ is a Whitney stratified set equipped with the following stratification:

$$\{f^{-1}(\delta)\cap V_\alpha\cap\mathring{B}_\epsilon, f^{-1}(\delta)\cap V_\alpha\cap S_\epsilon\mid V_\alpha\not\subseteq X^f\}.$$

Definition 3.1. *We call the set $f^{-1}(\delta)\cap X\cap B_\epsilon$, where $0<|\delta|\ll\epsilon\ll 1$, a real Milnor fibre of f.*

We will use the following notation: $M_f^{\delta,\epsilon}=f^{-1}(\delta)\cap X\cap B_\epsilon$.

Now we consider another subanalytic function $g:(X,0)\to(\mathbb{R},0)$ and we assume that it is the restriction to X of a C^2-subanalytic function $G:U\to$

$\mathbb{R}$. We denote by X^g the intersection $X \cap g^{-1}(0)$. Under some restrictions on g, we will study the topological behaviour of $g_{|M_f^{\delta,\epsilon}}$.

First we assume that g satisfies the following Condition (A):

- Condition (A): $g : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0.

This means that for each strata V_α of $\mathcal{V}$, $g : V_\alpha \setminus \{0\} \to \mathbb{R}$ is a submersion in a neighborhood of the origin.

In order to give the second assumption on g, we need to introduce some polar sets. Let V_α be a stratum of $\mathcal{V}$ not contained in X^f. Let $\Gamma_{f,g}^{V_\alpha}$ be the following set:

$$\Gamma_{f,g}^{V_\alpha} = \left\{ x \in V_\alpha \mid \operatorname{rank}[\nabla f_{|V_\alpha}(x), \nabla g_{|V_\alpha}(x)] < 2 \right\},$$

and let $\Gamma_{f,g}$ be the union $\cup \Gamma_{f,g}^{V_\alpha}$ where $V_\alpha \nsubseteq X^f$. We call $\Gamma_{f,g}$ the relative polar set of f and g with respect to the stratification $\mathcal{V}$. We will assume that g satifies the following Condition (B):

- Condition (B): the relative polar set $\Gamma_{f,g}$ is a 1-dimensional C^1-subanalytic set (possibly empty) in a neighborhood of the origin.

Note that Condition (B) implies that $\overline{\Gamma_{f,g}} \cap X^f \subset \{0\}$ in a neighborhood of the origin because the frontiers of the $\Gamma_{f,g}^{V_\alpha}$'s are 0-dimensional.

From Condition (A) and Condition (B), we can deduce the following result.

Lemma 3.2. *We have $\overline{\Gamma_{f,g}} \cap X^g \subset \{0\}$ in a neighborhood of the origin.*

Proof. If it is not the case then there is a C^1-subanalytic curve $\gamma : [0,\nu[\to \Gamma_{f,g} \cap X^g$ such that $\gamma(0) = 0$ and $\gamma(]0,\nu[) \subset X^g \setminus \{0\}$. We can also assume that $\gamma(]0,\nu[)$ is contained in a stratum V. For $t \in]0,\nu[$, we have

$$0 = (g \circ \gamma)'(t) = \langle \nabla g_{|V}(\gamma(t)), \gamma'(t) \rangle.$$

Since $\gamma(t)$ belongs to $\Gamma_{f,g}$ and $\nabla g_{|V}(\gamma(t))$ does not vanish for $g : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, we can conclude that $\langle \nabla f_{|V}(\gamma(t)), \gamma'(t) \rangle = 0$ and that $(f \circ \gamma)'(t) = 0$ for all $t \in]0,\nu[$. Therefore $f \circ \gamma \equiv 0$ because $f(0) = 0$ and $\gamma([0,\nu[)$ is included in X^f. This is impossible by the above remark. $\square$

Let $\mathcal{B}_1, \ldots, \mathcal{B}_l$ be the connected components of $\Gamma_{f,g}$, i.e. $\Gamma_{f,g} = \sqcup_{i=1}^l \mathcal{B}_i$. Each $\mathcal{B}_i$ is a C^1-subanalytic curve along which f is strictly increasing or decreasing and the intersection points of the $\mathcal{B}_i$'s with the fibre $M_f^{\delta,\epsilon}$ are exactly the critical points (in the stratified sense) of g on $X \cap f^{-1}(\delta) \cap \mathring{B}_\epsilon$. Let us write

$$M_f^{\delta,\epsilon} \cap \sqcup_{i=1}^l \mathcal{B}_i = \{p_1^{\delta,\epsilon}, \ldots, p_r^{\delta,\epsilon}\}.$$

Note that $r \leq l$.

Let us recall now the definition of the index of an isolated stratified critical point.

Definition 3.3. *Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set, equipped with a Whitney stratification. Let $p \in Z$ be an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, which is the restriction to Z of a C^2-subanalytic function Φ. We define the index of ϕ at p as follows :*

$$\mathrm{ind}(\phi, Z, p) = 1 - \chi\big(Z \cap \{\phi = \phi(p) - \eta\} \cap B_\epsilon(p)\big),$$

where $0 < \eta \ll \epsilon \ll 1$ and $B_\epsilon(p)$ is the closed ball of radius ϵ centered at p.

Our aim is to give a topological interpretation to the following sum:

$$\sum_{i=1}^{r} \mathrm{ind}(g, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}) + \mathrm{ind}(-g, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

For this, we will apply stratified Morse theory to $g_{|M_f^{\delta,\epsilon}}$. Note that the points p_i's are not the only critical points of $g_{|M_f^{\delta,\epsilon}}$ and other critical points can occur on the "boundary" $M_f^{\delta,\epsilon} \cap S_\epsilon$.

The next step is to study the behaviour of these "boundary" critical points for a generic choice of the distance function to the origin. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a C^2-subanalytic function which is a distance function to the origin. We denote by $\tilde{S}_\epsilon$ the level $\rho^{-1}(\epsilon)$ and by $\tilde{B}_\epsilon$ the set $\{\rho \leq \epsilon\}$. We will focus on the critical points of $g_{|X^f \cap \tilde{S}_\epsilon}$ and $g_{|X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon}$, with $0 < |\delta| \ll \epsilon \ll 1$.

For each stratum V of X^f, let

$$\Gamma_{g,\rho}^V = \left\{ x \in V \mid \mathrm{rank}[\nabla g_{|V}(x), \nabla \rho_{|V}(x)] < 2 \right\},$$

and let $\Gamma_{g,\rho}^{X^f} = \cup_{V \subset X^f} \Gamma_{g,\rho}^V$. By Lemma 2.4 and the fact that $g : (X^f, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, we can assume that $\Gamma_{g,\rho}^{X^f}$ is a C^1-subanalytic curve in a neighborhood of the origin.

Lemma 3.4. *We have $\Gamma_{g,\rho}^{X^f} \cap X^g \subset \{0\}$ in a neighborhood of the origin.*

Proof. Same proof as Lemma 3.2. □

Therefore if $\epsilon > 0$ is small enough, $g_{|\tilde{S}_\epsilon \cap X^f}$ has a finite number of critical points. They do not lie in the level $\{g = 0\}$ so by Lemma 2.3, they are outwards-pointing for $g_{|X^f \cap \tilde{B}_\epsilon}$ if they lie in $\{g > 0\}$ and inwards-pointing if they lie in $\{g < 0\}$.

Let us study now the critical points of $g_{|X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon}$. We will need the following lemma.

Lemma 3.5. *For every $\epsilon > 0$ sufficiently small, there exists $\delta(\epsilon) > 0$ such that for $0 < |\delta| \leq \delta(\epsilon)$, the points $p_i^{\delta,\epsilon}$ lie in $\tilde{B}_{\epsilon/4}$.*

Proof. Let

$$W = \left\{ (x, r, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \rho(x) = r, y = f(x) \text{ and } x \in \overline{\Gamma_{f,g}} \right\}.$$

Then W is a subanalytic set of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and since it is a graph over $\overline{\Gamma_{f,g}}$, its dimension is less or equal to 1. Let

$$\begin{array}{cccc} \pi & : & \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \times \mathbb{R} \\ & & (x,r,y) & \mapsto & (r,y), \end{array}$$

be the projection on the last two factors. Then $\pi_{|W} : W \to \pi(W)$ is proper and $\pi(W)$ is a closed subanalytic set in a neighborhood of the origin.

Let us write $Y_1 = \mathbb{R} \times \{0\}$ and let Y_2 be the closure of $\pi(W) \setminus Y_1$. Since Y_2 is a curve for W is a curve, 0 is isolated in $Y_1 \cap Y_2$. By Lojasiewicz's inequality, there exists a constant $C > 0$ and an integer $N > 0$ such that $|y| \geq Cr^N$ for (r,y) in Y_2 sufficiently close to the origin. So if $x \in \Gamma_{f,g}$ then $|f(x)| \geq C\rho(x)^N$ if $\rho(x)$ is small enough.

Let us fix $\epsilon > 0$ small. If $0 < |\delta| \leq \frac{1}{C}(\frac{\epsilon}{4})^N$ and $x \in f^{-1}(\delta) \cap \Gamma_{f,g}$ then $\rho(x) \leq \frac{\epsilon}{4}$. $\square$

For each stratum $V \not\subseteq X^f$, let

$$\Gamma^V_{f,g,\rho} = \left\{ x \in V \mid \operatorname{rank}[\nabla f_{|V}(x), \nabla g_{|V}(x), \nabla \rho_{|V}(x)] < 3 \right\},$$

and let $\Gamma_{f,g,\rho} = \cup_{V \not\subseteq X^f} \Gamma^V_{f,g,\rho}$. By Lemma 2.6, we can assume that $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ is a C^1-subanalytic manifold of dimension 2. Let us choose $\epsilon > 0$ small enough so that $\tilde{S}_\epsilon$ intersects $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ transversally. Therefore $(\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon$ is subanalytic curve. By Lemma 3.4, we can find $\delta(\epsilon) > 0$ such that $f^{-1}\big([\delta(\epsilon), -\delta(\epsilon)]\big) \cap \tilde{S}_\epsilon \cap \Gamma_{f,g}$ is empty and so

$$f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap (\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon = f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon.$$

Let $C_1, \ldots, C_t$ be the connected components of $f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon$ whose closure intersects $X^f \cap \tilde{S}_\epsilon$. Note that by Thom's (a_f)-condition, for each $i \in \{1, \ldots, t\}$, $\overline{C_i} \cap X^f$ is subset of $\Gamma^{X^f}_{g,\rho}$. Let z_i be a point in $\overline{C_i} \cap X^f$. Since $C_i \cap X^f = \emptyset$, there exists $0 < \delta'_i(\epsilon) \leq \delta(\epsilon)$ such that the fibre $f^{-1}(\delta)$, $0 < |\delta| \leq \delta'_i(\epsilon)$, intersects C_i transversally in a neighborhood of z_i.

Let us choose δ such that $0 < |\delta| \leq \operatorname{Min}\{\delta'_i(\epsilon) \mid i = 1, \ldots, t\}$. Then the fibre $f^{-1}(\delta)$ intersect the C_i's transversally and $f^{-1}(\delta) \cap (\cup_i C_i)$ is exactly the set of critical points of $g_{|f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$. We have proved:

Lemma 3.6. *For $0 < |\delta| \ll \epsilon \ll 1$, $g_{|f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$ has a finite number of critical points, which are exactly the points in $\Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon \cap f^{-1}(\delta)$.*

$\square$

Let $\{s_1^{\delta,\epsilon}, \ldots, s_u^{\delta,\epsilon}\}$ be the set of critical points of $g_{|f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$.

Lemma 3.7. *For $i \in \{1, \ldots, u\}$, $g(s_i^{\delta,\epsilon}) \neq 0$ and $s_i^{\delta,\epsilon}$ is outwards-pointing (resp. inwards-pointing) if and only if $g(s_i^{\delta,\epsilon}) > 0$ (resp. $g(s_i^{\delta,\epsilon}) < 0$).*

Proof. Note that $s_i^{\delta,\epsilon}$ is necessarily outwards-pointing or inwards-pointing because $s_i^{\delta,\epsilon} \notin \Gamma_{f,g}$.

Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta,\epsilon}$ such that $g(s_i^{\delta,\epsilon}) = 0$. Then we can construct a sequence of points $(\sigma_n)_{n\in\mathbb{N}}$ such that $g(\sigma_n) = 0$ and σ_n is a critical point of $g_{|f^{-1}(\frac{1}{n})\cap X\cap\tilde{S}_\epsilon}$. We can also assume that the points σ_n's belong to the same stratum S and that they tend to $\sigma \in V$ where $V \subseteq X^f$ and $V \subset \partial\overline{S}$. Therefore we have a decomposition:

$$\nabla g_{|S}(\sigma_n) = \lambda_n \nabla f_{|S}(\sigma_n) + \mu_n \nabla\rho_{|S}(\sigma_n).$$

Now by Whitney's condition (a), $T_{\sigma_n}S$ tends to a linear space T such that $T_\sigma V \subset T$. So $\nabla g_{|S}(\sigma_n)$ tends to a vector in T whose orthogonal projection on $T_\sigma V$ is exactly $\nabla g_{|V}(\sigma)$. Similarly $\nabla\rho_{|S}(\sigma_n)$ tends to a vector in T whose orthogonal projection on $T_\sigma V$ is exactly $\nabla\rho_{|V}(\sigma)$. By Thom's condition, $\nabla f_{|S}(\sigma_n)$ tends to a vector in T which is orthogonal to $T_\sigma V$, so we see that $\nabla g_{|V}(\sigma)$ and $\nabla\rho_{|V}(\sigma)$ are colinear which means that σ is a critical point of $g_{|X^f\cap\tilde{S}_\epsilon}$. But since $g(\sigma_n) = 0$, we find that $g(\sigma) = 0$, which is impossible by Lemma 3.4. This proves the first assertion.

To prove the second one, we use the same method. Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta,\epsilon}$ such that $g(s_i^{\delta,\epsilon}) > 0$ and $s_i^{\delta,\epsilon}$ is an inwards-pointing critical point for $g_{|X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon}$. Then we can construct a sequence of points $(\tau_n)_{n\in\mathbb{N}}$ such that $g(\tau_n) > 0$ and τ_n is an inwards-pointing critical point for $g_{|f^{-1}(\frac{1}{n})\cap X\cap\tilde{S}_\epsilon}$. We can also assume that the points τ_n's belong to the same stratum S and that they tend to $\tau \in V$ where $V \subseteq X^f$ and $V \subset \partial\overline{S}$. Therefore, we have a decomposition:

$$\nabla g_{|S}(\tau_n) = \lambda_n \nabla f_{|S}(\tau_n) + \mu_n \nabla\rho_{|S}(\tau_n),$$

with $\mu_n < 0$. Using the same arguments as above, we find that $\nabla g_{|V}(\tau) = \mu\nabla\rho_{|S}(\tau)$ with $\mu \leq 0$ and $g(\tau) \geq 0$. This contradicts the remark after Lemma 3.4. Of course, this proof works for $\delta < 0$. $\square$

Let $\Gamma_{g,\rho}$ be the following polar set:

$$\Gamma_{g,\rho} = \{x \in U \mid \operatorname{rank}[\nabla g(x), \nabla\rho(x)] < 2\}.$$

By Lemma 2.5 and Lemma 2.1, we can assume that $\Gamma_{g,\rho} \setminus \{g = 0\}$ does not intersect $X^f \setminus \{0\}$ in a neighborhood of 0 and so $\Gamma_{g,\rho} \setminus \{g = 0\}$ does not intersect $X^f \cap \tilde{S}_\epsilon$ for $\epsilon > 0$ sufficiently small. Since the critical points of $g_{|X^f\cap\tilde{S}_\epsilon}$ lie outside $\{g = 0\}$, they do not belong to $\Gamma_{g,\rho}\cap\tilde{S}_\epsilon$ and so the critical points of $g_{|f^{-1}(\delta)\cap X\cap\tilde{S}_\epsilon}$ do not neither if δ is sufficiently small. Hence at each critical point of $g_{|f^{-1}(\delta)\cap X\cap\tilde{S}_\epsilon}$, $g_{|\tilde{S}_\epsilon}$ is a submersion. We are in position to apply Theorem 3.1 and Lemma 2.1 in [15]. For $0 < |\delta| \ll \epsilon \ll 1$, we set

$$I(\delta,\epsilon,g) = \sum_{i=1}^{r} \operatorname{ind}(g, X\cap f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta,\epsilon,-g) = \sum_{i=1}^{r} \operatorname{ind}(-g, X\cap f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

Theorem 3.8. *We have*

$$I(\delta,\epsilon,g)+I(\delta,\epsilon,-g)=2\chi\big(X\cap f^{-1}(\delta)\cap\tilde{B}_\epsilon\big)$$
$$-\chi\big(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\big)-\chi\big(X^g\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\big).$$

Proof. Let us denote by $\{a_j^+\}_{j=1}^{\alpha^+}$ (resp. $\{a_j^-\}_{j=1}^{\alpha^-}$) the outwards-pointing (resp. inwards-pointing) critical points of $g: X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\to\mathbb{R}$. Applying Morse theory type theorem ([15], Theorem 3.1) and using Lemma 2.1 in [15], we can write

$$I(\delta,\epsilon,g)+\sum_{j=1}^{\alpha^-}\mathrm{ind}(g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)=\chi(X\cap f^{-1}(\delta)\cap\tilde{B}_\epsilon)\quad(1),$$

$$I(\delta,\epsilon,-g)+\sum_{j=1}^{\alpha^+}\mathrm{ind}(-g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,-a_j^+)=\chi(X\cap f^{-1}(\delta)\cap\tilde{B}_\epsilon)\ (2).$$

Let us evaluate

$$\sum_{j=1}^{\alpha^-}\mathrm{ind}(g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)+\sum_{j=1}^{\alpha^+}\mathrm{ind}(-g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+).$$

Since the outwards-pointing critical points of $g_{|X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon}$ lie in $\{g>0\}$ and the inwards-pointing critical points of $g_{|X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon}$ lie in $\{g<0\}$, we have

$$\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g\geq 0\})-\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\})=$$
$$\sum_{j=1}^{\alpha^+}\mathrm{ind}(g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,,a_j^+)\quad(3),$$

and

$$\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g\leq 0\})-\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\})=$$
$$\sum_{j=1}^{\alpha^-}\mathrm{ind}(-g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)\quad(4).$$

Therefore making (3) + (4) and using the Mayer-Vietoris sequence, we find

$$\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon)-\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\})=$$
$$\sum_{j=1}^{\alpha^+}\mathrm{ind}(g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+)+\sum_{j=1}^{\alpha^-}\mathrm{ind}(-g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)\quad(5).$$

Moreover we have

$$\chi(X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon)=\sum_{j=1}^{\alpha^+}\mathrm{ind}(g,X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+)$$

$$+\sum_{j=1}^{\alpha^-} \mathrm{ind}(g, X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon, a_j^-) \quad (6),$$

$$\chi(X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon) = \sum_{j=1}^{\alpha^+} \mathrm{ind}(-g, X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon, a_j^+)$$
$$+\sum_{j=1}^{\alpha^-} \mathrm{ind}(-g, X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon, a_j^-) \quad (7).$$

The combination $-(5)+(6)+(7)$ leads to

$$\chi(X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon) + \chi(X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon \cap \{g=0\}) =$$
$$\sum_{j=1}^{\alpha^+} \mathrm{ind}(-g, X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon, a_j^+) + \sum_{j=1}^{\alpha^-} \mathrm{ind}(g, X\cap f^{-1}(\delta)\cap \tilde{S}_\epsilon, a_j^-).$$

□

Let us assume now that $(X,0)$ is equipped with a Whitney stratification $\mathcal{W} = \cup_{\alpha\in A} W_\alpha$ and $f:(X,0)\to(\mathbb{R},0)$ has an isolated critical point at 0. In this situation, our results apply taking for $\mathcal{V}$ the following stratification:

$$\left\{W_\alpha \setminus f^{-1}(0), W_\alpha\cap f^{-1}(0)\setminus\{0\}, \{0\} \mid W_\alpha \in \mathcal{W}\right\}.$$

Corollary 3.9. *If $f:(X,0)\to(\mathbb{R},0)$ has an isolated stratified critical point at 0, then*

$$I(\delta,\epsilon,g) + I(\delta,\epsilon,-g) = 2\chi\big(X\cap f^{-1}(\delta)\cap \tilde{B}_\epsilon\big)$$
$$-\chi\big(X^f\cap \tilde{S}_\epsilon\big) - \chi\big(X^f\cap X^g\cap \tilde{S}_\epsilon\big).$$

Proof. For each stratum W of X, let

$$\Gamma^W_{f,\rho} = \left\{x\in W \mid \mathrm{rank}[\nabla f_{|W}(x), \nabla\rho_{|W}(x)] < 2\right\},$$

and let $\Gamma_{f,\rho} = \cup_W \Gamma^W_{f,\rho}$. By Lemma 3.4 applied to X and f instead of X^f and g, $\Gamma_{f,\rho}\cap\{f=0\}\subset\{0\}$ in a neighborhood of the origin and so 0 is a regular value of $f: X\cap\tilde{S}_\epsilon\to\mathbb{R}$ for ϵ sufficiently small. By Thom-Mather's second isotopy lemma, $X\cap f^{-1}(0)\cap\tilde{S}_\epsilon$ is homeomorphic to $X\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon$ for δ sufficiently small.

Now let p be a stratified critical point of $f: X^g\to\mathbb{R}$. By Lemma 2.1, we know that p belongs to $f^{-1}(0)\cap X^g$ and so p is also a critical point of $g: X^f\to\mathbb{R}$. Hence $p=0$ by Condition (A), and $f: X^g\to\mathbb{R}$ has an isolated stratified critical point at 0. As above, we conclude that $X^f\cap X^g\cap\tilde{S}_\epsilon$ is homeomorphic to $X^g\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon$. □

Let $\omega(x) = \sqrt{x_1^2+\cdots+x_n^2}$ be the euclidian distance to the origin. As explained by Durfee in [10], Lemma 1.8 and Lemma 3.6, there is a neighborhood Ω of 0 in $\mathbb{R}^n$ such that for every stratum V of X^f, $\nabla\omega_{|V}$ and $\nabla\rho_{|V}$ are non-zero and do not point in opposite direction in $\Omega\setminus\{0\}$. Applying

Durfee's argument ([10], Proposition 1.7 and Proposition 3.5), we see that $X^f \cap \tilde{S}_\epsilon$ is homeomorphic to $X^f \cap S_{\epsilon'}$ for $\epsilon, \epsilon' > 0$ sufficiently small. Similarly $X^f \cap X^g \cap \tilde{S}_\epsilon$ and $X^f \cap X^g \cap S_{\epsilon'}$ are homemorphic. Now let us compare $X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon$ and $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$. Let us choose ϵ' and ϵ such that

$$X \cap f^{-1}(\delta) \cap B_{\epsilon'} \subset X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon \subset \Omega.$$

If δ is sufficiently small then, for every stratum $V \not\subseteq X^f$, $\nabla \omega_{|V \cap f^{-1}(\delta)}$ and $\nabla \rho_{|V \cap f^{-1}(\delta)}$ are non-zero and do not point in opposite direction in $\tilde{B}_\epsilon \setminus \mathring{B}_{\epsilon'}$. Otherwise, by Thom's (a_f)-condition, we would find a point p in $X^f \cap (\tilde{B}_\epsilon \setminus \mathring{B}_{\epsilon'})$ such that either $\nabla \omega_{|S}(p)$ or $\nabla \rho_{|S}(p)$ vanish or $\nabla \omega_{|S}(p)$ and $\nabla \rho_{|S}(p)$ point in opposite direction, where S is the stratum of X^f that contains p. This is impossible if we are sufficiently close to the origin. Now, applying the same arguments as Durfee [10], Proposition 1.7 and Proposition 3.5, we see that $X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$ and that $X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap S_{\epsilon'}$.

Theorem 3.10. *We have*

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(X \cap f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon).$$

□

Corollary 3.11. *If $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, then*

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(\mathrm{Lk}(X^f)) - \chi(\mathrm{Lk}(X^f \cap X^g)).$$

□

Let us remark if dim $X = 2$ then in Theorem 3.10 and in Corollary 3.11, the last term of the right-hand side of the equality vanishes. If dim $X = 1$ then in Theorem 3.10 and in Corollary 3.11, the last two terms of the right-hand side of the equality vanish.

4. An infinitesimal Gauss-Bonnet formula

In this section, we apply the results of the previous section to the case of linear forms and we establish a Gauss-Bonnet type formula for the real Milnor fibre.

We will first show that generic linear forms satisfy Condition (A) and Condition (B). For $v \in S^{n-1}$, let us denote by v^* the function $v^*(x) = \langle v, x \rangle$.

Lemma 4.1. *There exists a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_1$, $\{v^* = 0\}$ intersects $X \setminus \{0\}$ transversally (in the stratified sense) in a neighborhood of the origin.*

Proof. It is a particular case of Lemma 3.8 in [14]. □

Corollary 4.2. *If $v \notin \Sigma_1$ then $v^*_{|X} : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified point at 0.*

Proof. By Lemma 2.1, we know that the stratified critical points of $v^*_{|X}$ lie in $\{v^* = 0\}$. But since $\{v^* = 0\}$ intersects $X \setminus \{0\}$ transversally, the only possible critical point of $v^*_{|X} : (X, 0) \to (\mathbb{R}, 0)$ is the origin. $\square$

Lemma 4.3. *There exists a subanalytic set $\Sigma_2 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_2$, then Γ_{f,v^*} is a C^1-subanalytic curve (possibly empty) in a neighborhood of* 0.

Proof. Let V be stratum of dimension e such that $V \not\subseteq X^f$. We can assume that $e \geq 2$. Let

$$M_V = \left\{ (x,y) \in V \times \mathbb{R}^n \mid \operatorname{rank}[\nabla f_{|V}(x), \nabla y^*_{|V}(x)] < 2 \right\}.$$

It is a subanalytic manifold of class C^1 and of dimension $n+1$. To see this, let us pick a point (x,y) in M_V. In a neighborhood of x, V is defined by the vanishing of $k = n - e$ C^2-subanalytic functions $f_1, \ldots, f_k$. Since V is not included in X^f, $f : V \to \mathbb{R}$ is a submersion and we can assume that in a neighborhood of x, the following $(k+1) \times (k+1)$-minor:

$$\frac{\partial(f_1, \ldots, f_k, f)}{\partial(x_1, \ldots, x_k, x_{k+1})},$$

does not vanish. Therefore, in a neighborhood of (x,y), M_V is defined by the vanishing of the following $(k+2) \times (k+2)$-minors:

$$\frac{\partial(f_1, \ldots, f_k, f, y^*)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_{k+2})}, \ldots, \frac{\partial(f_1, \ldots, f_k, f, y^*)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_n)}.$$

A simple computation of determinants shows that the gradient vectors of these minors are linearly independent. As in previous lemmas, we show that Σ_{f,v^*} is one-dimensional considering the projection

$$\begin{array}{cccc} \pi_2 & : & M^V & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y. \end{array}$$

Since $\Gamma_{f,v^*} = \cup_{V \not\subseteq X^f} \Gamma^V_{f,v^*}$, we get the result. $\square$

Let $\Sigma = \Sigma_1 \cup \Sigma_2$, it is a subanalytic subset of S^{n-1} of positive codimension and if $v \notin \Sigma$ then v^* satisfies Conditions (A) and (B). In particular, $v^*_{|f^{-1}(\delta) \cap X \cap B_\epsilon}$ has a finite number of critical points $p_1^{\delta,\epsilon}, \ldots, p_{r_v}^{\delta,\epsilon}$. We recall that

$$I(\delta, \epsilon, v^*) = \sum_{i=1}^{r_v} \operatorname{ind}(v^*, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta, \epsilon, -v^*) = \sum_{i=1}^{r_v} \operatorname{ind}(-v^*, X \cap f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

In this situation, Theorem 3.10 and Corollary 3.11 become

Corollary 4.4. *If $v \notin \Sigma$ then*

$$I(\delta,\epsilon,v^*)+I(\delta,\epsilon,-v^*) = 2\chi(M_f^{\delta,\epsilon})-\chi(X\cap f^{-1}(\delta)\cap S_\epsilon)-\chi(X^{v^*}\cap f^{-1}(\delta)\cap S_\epsilon).$$

Furthermore, if $f:(X,0)\to(\mathbb{R},0)$ has an isolated stratified critical point at 0, then

$$I(\delta,\epsilon,v^*)+I(\delta,\epsilon,-v^*) = 2\chi(M_f^{\delta,\epsilon})-\chi(\mathrm{Lk}(X^f))-\chi(\mathrm{Lk}(X^f\cap X^{v^*})).$$

□

As an application, we give a Gauss-Bonnet formula for the Milnor fibre $M_f^{\delta,\epsilon}$. Let $\Lambda_0(X\cap f^{-1}(\delta),-)$ be the Gauss-Bonnet measure on $X\cap f^{-1}(\delta)$ defined by

$$\Lambda_0(X\cap f^{-1}(\delta),U') = \frac{1}{s_{n-1}}\int_{S^{n-1}}\sum_{x\in U'}\mathrm{ind}(v^*,X\cap f^{-1}(\delta),x)dx,$$

where U' is a Borel set of $X\cap f^{-1}(\delta)$ (see [6], page 299) and s_{n-1} is the volume of the unit sphere S^{n-1}. Note that if x is not a critical point of $v^*_{|X\cap f^{-1}(\delta)}$ then $\mathrm{ind}(v^*,X\cap f^{-1}(\delta),x)=0$. We are going to evaluate

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Lambda_0(X\cap f^{-1}(\delta),M_f^{\delta,\epsilon}).$$

Theorem 4.5. *We have*

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Lambda_0(X\cap f^{-1}(\delta),M_f^{\delta,\epsilon}) = \chi(M_f^{\delta,\epsilon})-\frac{1}{2}\chi(X\cap f^{-1}(\delta)\cap S_\epsilon) - \frac{1}{2s_{n-1}}\int_{S^{n-1}}\chi(X\cap f^{-1}(\delta)\cap\{v^*=0\}\cap S_\epsilon)dv.$$

Furthermore, if $f:(X,0)\to(\mathbb{R},0)$ has an isolated stratified critical point at 0, then

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Lambda_0(X\cap f^{-1}(\delta),M_f^{\delta,\epsilon}) = \chi(M_f^{\delta,\epsilon})-\frac{1}{2}\chi(\mathrm{Lk}(X^f)) - \frac{1}{2s_{n-1}}\int_{S^{n-1}}\chi(\mathrm{Lk}(X^f\cap X^{v^*}))dv.$$

Proof. By definition, we have

$$\Lambda_0(X\cap f^{-1}(\delta),M_f^{\delta,\epsilon}) = \frac{1}{s_{n-1}}\int_{S^{n-1}}\sum_{x\in M_f^{\delta,\epsilon}}\mathrm{ind}(v^*,X\cap f^{-1}(\delta),x)dv.$$

It is not difficult to see that

$$\Lambda_0(X\cap f^{-1}(\delta),M_f^{\delta,\epsilon}) = \frac{1}{2s_{n-1}}\int_{S^{n-1}}\Big[\sum_{x\in M_f^{\delta,\epsilon}}\mathrm{ind}(v^*,X\cap f^{-1}(\delta),x)+\mathrm{ind}(-v^*,X\cap f^{-1}(\delta),x)\Big]dv.$$

Note that if $v \notin \Sigma$ then

$$\sum_{x \in M_f^{\delta,\epsilon}} \mathrm{ind}(v^*, X \cap f^{-1}(\delta), x) + \mathrm{ind}(-v^*, X \cap f^{-1}(\delta), x)$$

is equal to $I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*)$ and is uniformly bounded by Hardt's theorem. By Lebesgue's theorem, we obtain

$$\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta,\epsilon}) =$$

$$\frac{1}{2s_{n-1}} \int_{S^{n-1}} \lim_{\epsilon \to 0} \lim_{\delta \to 0} [I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*)] dv.$$

We just have to apply the previous corollary to conclude. □

5. Infinitesimal linear kinematic formulas

In this section, we apply the results of the previous section to the case of a linear function in order to obtain "infinitesimal" linear kinematic formulas for closed subanalytic germs.

We start recalling known facts on the geometry of subanalytic sets. We need some notations:

- for $k \in \{0, \ldots, n\}$, G_n^k is the Grassmann manifold of k-dimensional linear subspaces in $\mathbb{R}^n$ and g_n^k is its volume,
- for $k \in \mathbb{N}$, b_k is the volume of the k-dimensional unit ball and s_k is the volume of the k-dimensional unit sphere.

In [17], Fu developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set $X \subset \mathbb{R}^n$ a sequence of curvature measures

$$\Lambda_0(X, -), \ldots, \Lambda_n(X, -),$$

called the Lipschitz-Killing measures. He proved several integral geometry formulas, among them a Gauss-Bonnet formula and a kinematic formula. Later another description of the measures using stratified Morse theory was given by Broecker and Kuppe [6] (see also [5]). The reader can refer to [14], Section 2, for a rather complete presentation of these two approaches and for the definition of the Lipschitz-Killing measures.

Let us give some comments on these Lipschitz-Killing curvatures. If dim $X = d$ then

$$\Lambda_{d+1}(X, U') = \cdots = \Lambda_n(X, U') = 0,$$

for any Borel set U' of X and $\Lambda_d(X, U') = \mathcal{L}_d(U')$, where $\mathcal{L}_d$ is the d-dimensional Lebesgue measure in $\mathbb{R}^n$. Furthemore if X is smooth then for any Borel set U' of X and for $k \in \{0, \ldots, d\}$, $\Lambda_k(X, U')$ is related to the classical Lipschitz-Killing-Weil curvature K_{d-k} through the following equality:

$$\Lambda_k(X, U') = \frac{1}{s_{n-d-k-1}} \int_{U'} K_{d-k}(x) dx.$$

In [14], Section 5, we studied the asymptotic behaviour of the Lipschitz-Killing measures in the neighborhood of a point of X. Namely we proved the following theorem ([14], Theorem 5.1).

Theorem 5.1. *Let $X \subset \mathbb{R}^n$ be a closed subanalytic set such that $0 \in X$. We have:*

$$\lim_{\epsilon \to 0} \Lambda_0(X, X \cap B_\epsilon) = 1 - \frac{1}{2}\chi(\mathrm{Lk}(X)) - \frac{1}{2g_n^{n-1}} \int_{G_n^{n-1}} \chi(\mathrm{Lk}(X \cap H))dH.$$

Furthermore for $k \in \{1, \ldots, n-2\}$, we have:

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X \cap H))dH + \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L))dL,$$

and:

$$\lim_{\epsilon \to 0} \frac{\Lambda_{n-1}(X, X \cap B_\epsilon)}{b_{n-1}\epsilon^{n-1}} = \frac{1}{2g_n^2} \int_{G_n^2} \chi(\mathrm{Lk}(X \cap H))dH,$$

$$\lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_\epsilon)}{b_n \epsilon^n} = \frac{1}{2g_n^1} \int_{G_n^1} \chi(\mathrm{Lk}(X \cap H))dH.$$

In the sequel, we will use these equalities and Theorem 4.5 to establish linear kinematic types formulas for the quantities $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k}$, $k = 1, \ldots, n$. Let us start with some lemmas. We work with a closed subanalytic set X such that $0 \in X$, equipped with a Whitney stratification $\{W_\alpha\}_{\alpha \in A}$.

Lemma 5.2. *Let f be a C^2-subanalytic function such that $f_{|X} : X \to \mathbb{R}$ has an isolated stratified critical point at 0. Then for $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi(M_f^{\delta,\epsilon}) + \chi(M_f^{-\delta,\epsilon}) = \chi(\mathrm{Lk}(X)) + \chi(\mathrm{Lk}(X^f)).$$

Proof. With the same technics and arguments as the ones we used in order to establish Corollary 3.11, we can prove that

$$\mathrm{ind}(f, X, 0) + \mathrm{ind}(-f, X, 0) = 2\chi(X \cap B_\epsilon) - \chi(\mathrm{Lk}(X)) - \chi(\mathrm{Lk}(X^f)).$$

We conclude thanks to the following equalities

$$\mathrm{ind}(f, X, 0) = 1 - \chi(M_f^{-\delta,\epsilon}), \ \mathrm{ind}(-f, X, 0) = 1 - \chi(M_f^{\delta,\epsilon}),$$

and

$$\chi(X \cap B_\epsilon) = 1.$$

□

Corollary 5.3. *There exist a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma$ then for $0 < \delta \ll \epsilon \ll 1$,*

$$\chi(M_{v^*}^{\delta,\epsilon}) + \chi(M_{v^*}^{-\delta,\epsilon}) = \chi(\mathrm{Lk}(X)) + \chi(\mathrm{Lk}(X \cap \{v^* = 0\})).$$

Proof. Apply Corollary 4.2 and Lemma 5.2. □

Lemma 5.4. *Let $S \subset \mathbb{R}^n$ be C^2-subanalytic manifold. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$ and let $G^1_{H^\perp}$ be the Grassmann manifold of lines in the orthogonal complement $H^\perp$ of H. There exists a subanalytic set $\Sigma'_H \subset G^1_{H^\perp}$ of positive codimension such that if $\nu \notin \Sigma'_H$ then $H \oplus \nu$ intersects $S \setminus \{0\}$ transversally.*

Proof. Assume that S has dimension e and that H is given by the equations $x_1 = \ldots = x_k = 0$ so that $H^\perp = \mathbb{R}^k$ with coordinate system $(x_1, \ldots, x_k)$. Let W be defined by

$$W = \Big\{(x, v_1, \ldots, v_{k-1}) \in \mathbb{R}^n \times (\mathbb{R}^k)^{k-1} \mid x \in S \setminus \{0\} \\ \text{and } \langle x, v_1\rangle = \cdots = \langle x, v_{k-1}\rangle = 0\Big\},$$

where $v_i \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Let us show that W is a C^2-subanalytic manifold of dimension $e + (k-1)^2$. Let (y, w) be a point in W. We can assume that around y, S is defined by the vanishing of $n - e$ C^2-subanalytic functions $f_1, \ldots, f_{n-e}$. Hence in a neighborhood of (y, w), W is defined by the equations:

$$f_1(x) = \ldots = f_{n-e}(x) = 0 \text{ and } \langle x, v_1\rangle = \cdots = \langle x, v_{k-1}\rangle = 0.$$

Because $y \neq 0$, we see that the gradient vectors of this $n-e+k-1$ functions are linearly independent at (y, w). This enables us to conclude that W is a C^2-subanalytic manifold of dimension $e + (k-1)^2$. Let π_2 be the following projection:

$$\pi_2 : W \to (\mathbb{R}^n)^{n-k}, (x, v_1, \ldots, v_{n-k}) \mapsto (v_1, \ldots, v_{n-k}).$$

Bertini-Sard's theorem implies that the set of critical values of π_2 is a subanalytic set of positive codimension. If $(v_1, \ldots, v_{k-1})$ lies outside this subanalytic set then the $(n-k+1)$-plane $\{x \in \mathbb{R}^n \mid \langle x, v_1\rangle = \cdots = \langle x, v_{k-1}\rangle = 0\}$ contains H and intersects $S \setminus \{0\}$ transversally. $\square$

Now we can present our infinitesimal linear kinematic formulas. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, and let $S^{k-1}_{H^\perp}$ be the unit sphere of the orthogonal complement of H. Let v be an element in $S^{k-1}_{H^\perp}$. For $\delta > 0$, we denote by $H_{v,\delta}$ the $(n-k)$-dimensional affine space $H + \delta v$ and we set

$$\beta_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(H_{\delta,v} \cap X, H_{\delta,v} \cap X \cap B_\epsilon).$$

Then we set

$$\beta_0(H) = \frac{1}{s_{k-1}} \int_{S^{k-1}_{H^\perp}} \beta_0(H, v) dv.$$

Theorem 5.5. *For $k \in \{1, \ldots, n\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \beta_0(H) dH.$$

Proof. We treat first the case $k \in \{1,\dots,n-2\}$. By Theorem 5.1 , we know that

$$\lim_{\epsilon\to 0}\frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k\epsilon^k} = -\frac{1}{2g_n^{n-k-1}}\int_{G_n^{n-k-1}}\chi(\mathrm{Lk}(X\cap H))dH + \frac{1}{2g_n^{n-k+1}}\int_{G_n^{n-k+1}}\chi(\mathrm{Lk}(X\cap L))dL.$$

By Lemma 3.8 in [14], we know that generically H intersects $X\setminus\{0\}$ transversally in a neighborhood of the origin. Let us fix H that satisfies this generic property. For any $v \in S_{H^\perp}^{k-1}$, let ν be the line generated by v and let L_v be the $(n-k+1)$-plane defined by $L_v = H \oplus \nu$. By Lemma 5.4, we know that for v generic in $S_{H^\perp}^{k-1}$, L_v intersects $X \setminus \{0\}$ transversally in a neighborhood of the origin. Therefore, $v^*_{|X\cap L_v}$ has an isolated singular point at 0 and we can apply Theorem 4.5. We have

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Lambda_0(X\cap L_v\cap\{v^*=\delta\}, X\cap L_v\cap\{v^*=\delta\}\cap B_\epsilon) =\\ \chi(X\cap L_v\cap\{v^*=\delta\}\cap B_\epsilon) - \frac{1}{2}\chi(\mathrm{Lk}(X\cap L_v\cap\{v^*=0\}))\\ -\frac{1}{2s_{n-k}}\int_{S_{L_v}^{n-k}}\chi(\mathrm{Lk}(X\cap L_v\cap\{v^*=0\}\cap\{w^*=0\}))dw,\end{aligned}$$

where $S_{L_v}^{n-k}$ is the unit sphere of L_v. Let us remark that $L_v\cap\{v^*=\delta\}$ is exactly $H_{v,\delta}$ and that $L_v\cap\{v^*=0\}$ is H. We can also apply Lemma 5.2 to $v^*_{|X\cap L_v}$ to obtain the following relation:

$$\beta_0(H,v)+\beta_0(H,-v) = \chi(\mathrm{Lk}(X\cap L_v)) - \frac{1}{s_{n-k}}\int_{S_{L_v}^{n-k}}\chi(\mathrm{Lk}(X\cap H\cap\{w^*=0\}))dw.$$

Since $\beta(H)$ is equal to

$$\frac{1}{2s_{k-1}}\int_{S_{H^\perp}^{k-1}}\left[\beta_0(H,v)+\beta_0(H,-v)\right]dv,$$

we find that

$$\beta(H) = \frac{1}{2s_{k-1}}\int_{S_{H^\perp}^{k-1}}\chi(\mathrm{Lk}(X\cap L_v))dv - \frac{1}{2s_{k-1}s_{n-k}}\int_{S_{H^\perp}^{k-1}}\int_{S_{L_v}^{n-k}}\chi(\mathrm{Lk}(X\cap H\cap\{w^*=0\}))dwdv.$$

Replacing spheres with Grassman manifolds in this equality, we obtain

$$\beta(H) = \frac{1}{2g_k^1}\int_{G_{H^\perp}^1}\chi(\mathrm{Lk}(X\cap H\oplus\nu))d\nu - \frac{1}{2g_k^1 g_{n-k+1}^{n-k}}\int_{G_{H^\perp}^1}\int_{G_{H\oplus\nu}^{n-k}}\chi(\mathrm{Lk}(X\cap H\cap K))dKd\nu.$$

Therefore, we have

$$\frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \beta(H) dH = \frac{1}{2g_k^1 g_n^{n-k}} \int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \chi(\mathrm{Lk}(X \cap H \oplus \nu)) d\nu dH -$$

$$\frac{1}{2g_n^{n-k} g_k^1 g_{n-k+1}^{n-k}} \int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \int_{G^{n-k}_{H\oplus\nu}} \chi(\mathrm{Lk}(X \cap H \cap K)) dK d\nu dH.$$

Let us compute

$$\mathcal{I} = \frac{1}{2g_n^{n-k} g_k^1} \int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \chi(\mathrm{Lk}(X \cap H \oplus \nu)) d\nu dH.$$

Let $\mathcal{H}$ be the flag variety of pairs (L, H), $L \in G_n^{n-k+1}$ and $H \in G_L^{n-k}$. This variety is a bundle over G_n^{n-k}, each fibre being a G_k^1. Hence we have

$$\int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \chi(\mathrm{Lk}(X \cap H \oplus \nu)) d\nu dH = \int_{G_n^{n-k+1}} \int_{G_L^{n-k}} \chi(\mathrm{Lk}(X \cap L)) dH dL =$$

$$g_{n-k+1}^{n-k} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L)) dL.$$

Finally, we get that

$$\mathcal{I} = \frac{g_{n-k+1}^{n-k}}{2g_n^{n-k} g_k^1} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L)) dL =$$

$$\frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L)) dL.$$

Let us compute now

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k+1}^{n-k}} \int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \int_{G^{n-k}_{H\oplus\nu}} \chi(\mathrm{Lk}(X \cap H \cap K)) dK d\nu dH.$$

First, as we have just done above, we can write

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k+1}^{n-k}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k}} \int_{G_L^{n-k}} \chi(\mathrm{Lk}(X \cap H \cap K)) dK dH dL.$$

Then we remark (see [14], Corollary 3.11 for a similar argument) that

$$\frac{1}{g_{n-k+1}^{n-k}} \int_{G_L^{n-k}} \chi(\mathrm{Lk}(X \cap H \cap K)) dK = \frac{1}{g_{n-k}^{n-k-1}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ,$$

and so

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ dH dL.$$

Considering the flag variety of pairs (H, J), $H \in G_L^{n-k}$ and $J \in G_H^{n-k-1}$, and proceeding as above, we find

$$\int_{G_L^{n-k}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ dH = g_2^1 \int_{G_L^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ,$$

so

$$\mathcal{J} = \frac{g_2^1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ.$$

To finish the computation, we consider the flag variety of pairs (L, J), $L \in G_n^{n-k+1}$ and $J \in G_L^{n-k-1}$. It is a bundle over G_n^{n-k-1}, each fibre being a G_{k+1}^2. Hence we have

$$\mathcal{J} = \frac{g_2^1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k-1}} \int_{G^2_{J^\perp}} \chi(\mathrm{Lk}(X \cap J)) dJdM,$$

$$\mathcal{J} = \frac{g_2^1 g_{k+1}^2}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ =$$

$$\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X \cap J)) dJ.$$

This ends the proof for the case $k \in \{1, \ldots, n-2\}$. For $k = n-1$ or n, the proof is the same. We just have to remark that in these cases

$$\beta_0(H, v) + \beta_0(H, -v) = \chi(\mathrm{Lk}(X \cap L_v)),$$

and if $k = n-1$, dim $L_v = 2$ and if $k = n$, dim $L_v = 1$. $\square$

Let us end with some remarks on the limits $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k}$. We already know that if dim $X = d$ then $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = 0$ for $k \geq d+1$. This is also the case if $l < d_0$, where d_0 is the dimension of the stratum that contains 0. To see this let us first relate the limits $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k}$ to the polar invariants defined by Comte and Merle in [9]. They can be defined as follows. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, and let v be an element in $S_{H^\perp}^{k-1}$. For $\delta > 0$, we set

$$\lambda_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \chi(H_{\delta, v} \cap X \cap B_\epsilon),$$

and then

$$\sigma_k(X, 0) = \frac{1}{s_{k-1}} \int_{S_{H^\perp}^{k-1}} \lambda_0(H, v) dv.$$

Moreover, we put $\sigma_0(X, 0) = 1$.

Theorem 5.6. *For $k \in \{0, \ldots, n-1\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \sigma_k(X, 0) - \sigma_{k+1}(X, 0).$$

Furthermore, we have

$$\lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_\epsilon)}{b_n \epsilon^n} = \sigma_n(X, 0).$$

Proof. It is the same proof as Theorem 5.5. For example if $k \in \{0, \ldots, n-1\}$, we just have to remark that

$$\lambda_0(H, v) + \lambda_0(H, -v) = \chi(\mathrm{Lk}(X \cap L_v)) + \chi(\mathrm{Lk}(X \cap H)),$$

by Lemma 5.2, which implies that

$$\sigma_k(X, 0) = \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L))dL + \frac{1}{2g_n^{n-k}} \int_{G_n^{n-k}} \chi(\mathrm{Lk}(X \cap H))dH.$$

$\square$

It is explained in [9] that $\sigma_k(X, 0) = 1$ if $0 \leq k \leq d_0$, so if $k < d_0$ then $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = 0$.

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